7.8 IMPROPER INTEGRALS

ESSENTIAL QUESTION: What conditions make a definite integral "improper"?

Improper integrals occur when

- one (or both) limit of integration is ∞.

$$\int_1^{\infty} \frac{1}{x}\,dx$$

- there is a discontinuity in the integrand somewhere between the limits of integration.

$$\int_1^{3} \frac{5}{x-2}\,dx$$

CONTINUOUS FUNCTIONS

1. $f(x)$ is continuous on [a, ∞)⇨

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^{b} f(x)\,dx$$

Example:

$$\int_0^{\infty} xe^{-x}\,dx$$

*If the limit exists, the integral is said to ***converge***.

2. If f(x) is continuous on (-∞, b], then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^{b} f(x)\,dx$$

Example:

$$\int_{-\infty}^{0} \sin\frac{x}{2}\,dx$$

*If the limit does not exist, the integral is said to ***diverge***.

3. If f is continuous on (-∞, ∞), then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

$$= \lim_{a\to-\infty}\int_a^{c} f(x)\,dx + \lim_{b\to\infty}\int_c^{b} f(x)\,dx$$

Example:

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$$

DISCONTINUOUS FUNCTIONS

1. If f is continuous on [a, b), then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

Example: $\int_0^{\pi/2} \tan x\,dx$

2. If f is continuous on (a, b], then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$

Example: $\int_0^2 \frac{1}{x^2}\,dx$

3. If f is continuous on [a, b] except at x = c, where there is an infinite discontinuity, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{c \to d^-} \int_a^d f(x)\,dx + \lim_{c \to d^+} \int_d^b f(x)\,dx$$

Example: $\int_0^2 \frac{1}{(x-1)^2}\,dx$