5-2 Integration & Natural Logs

ESSENTIAL QUESTION: How are standard integration tecniques (especially u-sub) used to find integrals involving natural logs?

$$\int \frac{1}{x}dx = \ln|x| + C$$

In addition, if u is a differentiable function of x,

$$\int \frac{1}{u}du = \ln|u| + C$$

Examples:

1. $\int \frac{x}{x^2+1}dx$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\int \frac{1}{x^2+1} \cdot x\,dx$

$\frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + c$

$= \frac{1}{2}\ln|x^2+1| + C$

$\int \frac{3x^2}{x} \neq \int 3x - \frac{2}{x}$

Use long division to simplify the integrand when the degree of the numerator is greater than or equal to the degree of the denominator.

2. $\int \frac{2x^2+7x-3}{x-2}dx$

$$\begin{array}{r} 2x + 11 \\ x-2\,\overline{)\,2x^2+7x-3} \\ -(2x^2-4x) \\ 11x-3 \\ -(11x-22) \\ 19 \end{array}$$

$2x + 11 + \frac{19}{x-2}$

$= \int \left(2x+11+\frac{19}{x-2}\right)dx$

$u = x-2 \quad du = dx$

$\frac{2x^2}{2} + 11x + 19\int \frac{1}{x-2}dx$

$x^2 + 11x + 19 \cdot \ln|u| + c$

$x^2 + 11x + 19\ln|x-2| + C$

This one has a twist!

3. $\int \frac{1}{1+\sqrt{2x}}dx$

$u = 1+\sqrt{2x} \rightarrow \sqrt{2x} = u-1$

$du = \frac{1}{2\sqrt{2x}} \cdot 2\,dx$

$\sqrt{2x}\,du = dx$

$(u-1)\,du = dx$

$\int \frac{1}{u} \cdot (u-1)\,du$

$\int \left(\frac{u-1}{u}\right)du$

$\int \left(1 - \frac{1}{u}\right)du = u - \ln|u| + c_1$

$= 1 + \sqrt{2x} - \ln|1+\sqrt{2x}| + c_1$

$= \sqrt{2x} - \ln|1+\sqrt{2x}| + C$

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4. Solve the differential equation $\frac{dy}{dx} = \frac{\ln x}{x}$ if $y(1) = -2$.

$\int dy = \int \frac{\ln x}{x}dx$

$u = \ln x$

$du = \frac{1}{x}dx$

$y = \int \ln x \cdot \frac{1}{x}dx$

$y = \int u\,du$

$y = \frac{u^2}{2} + c \rightarrow y = \frac{1}{2}(\ln x)^2 + c$

$-2 = \frac{1}{2}(\ln 1)^2 + c$

$-2 = \frac{1}{2}(0)^2 + c$

$y = \frac{1}{2}(\ln x)^2 - 2$

COMPLETE LIST OF TRIG INTEGRALS

1. $\int \sin x\,dx = -\cos x + C$
2. $\int \cos x\,dx = \sin x + C$
3. $\int \tan x\,dx = -\ln|\cos x| + C$
4. $\int \cot x\,dx = \ln|\sin x| + C$
5. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
6. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
7. $\int \sec^2 x\,dx = \tan x + C$
8. $\int \csc^2 x\,dx = -\cot x + C$
9. $\int \sec x \tan x\,dx = \sec x + C$
10. $\int \csc x \cot x\,dx = -\csc x + C$

Examples:

1. $\int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x}dx = \int_{-\pi/4}^{\pi/4} \frac{1-\cos^2 x - \cos^2 x}{\cos x}dx$

$= \int_{-\pi/4}^{\pi/4} \left(\frac{1 - 2\cos^2 x}{\cos x}\right)dx = \int_{-\pi/4}^{\pi/4} (\sec x - 2\cos x)\,dx$

$= \ln|\sec x + \tan x| - 2\sin x + C$

Solve the differential equation $\frac{dr}{dt} = \frac{\sec^2 t}{\tan t + 1}$ if $r(\pi) = 4$.

$\int dr = \int \frac{\sec^2 t}{\tan t + 1}dt$

$u = \tan t + 1$

$du = \sec^2 t\,dt$

$r = \int \frac{1}{\tan t + 1} \cdot \sec^2 t\,dt$

$r = \int \frac{1}{u}du = \ln|u| + C$

$r = \ln|\tan t + 1| + C$

$4 = \ln|\tan \pi + 1| + C$

$4 = \ln 1 + C \rightarrow C = 4$

$\boxed{r = \ln|\tan t + 1| + 4}$