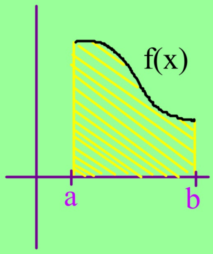
4.4 The Fundamental Theorem of Calculus

ESSENTIAL QUESTIONS

How can we

1. evaluate a definite integral using the Fundamental Theorem of Calculus?
2. apply the Mean Value Theorem for Integrals?
3. find the average value of a function on a closed interval?
4. understand and use the Second Fundamental Theorem of Calculus?

We've learned that we can express the area of a plane region as a definite integral:



$$A = \int_a^b f(x)dx$$

Problem: How do we evaluate the definite integral so we can find the exact area?

Fundamental Theorem of Calculus

If f is continuous on [a, b] and F is an antiderivative of f on [a, b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Evaluate $\int_3^6 x^2 dx$

Notice how the "+C" drops out. For this reason, we don't have to consider the "+C" when we evaluate definite integrals.

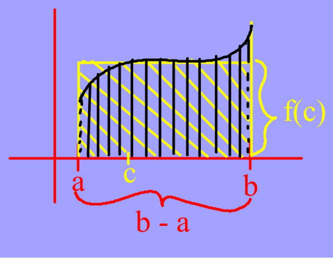
Evaluate $\int_9^{25} 4\sqrt{x}dx$

Find the area of the region bounded by the graphs of y = sin x, x = 0, x = π, and y = 0.

Mean Value Theorem for Integrals (MVTI)

If *f* is continuous on [a, b], then there exists a number *c* in [a, b] such that

$$\int_a^b f(x)dx = f(c)\cdot(b-a)$$



Example: Find the value of c guaranteed by the MVTI for $y = x^3 + 4$ on the interval [0, 4].

The value of f(c) (y-value) is called the average value of the function f(x).

We can solve the MVTI equation for f(c):

$$\int_a^b f(x)dx = f(c)\cdot(b-a)$$

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c)$$

As a result, we have the definition of the average value of a function:

Average value of a function:
Let f(c) be the average value of f(x) on [a, b]. Then

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Example: Find the average value of f(x) = cos x on the interval [0, π/2].

When we defined definite integrals, the limits of integration were constants.

$$\int_a^b f(x)dx$$

a and b are constants

f is a function of x

We can also look at definite integral in which the limits are functions of x.

$$F(x) = \int_a^x f(t)\,dt$$

variable

F is a funtion of x.

constant

f is a function of t.

Evaluate $F(x) = \int_1^x \sqrt{t}\,dt$

Now evaluate $F'(x) = \frac{d}{dx}\left[\int_1^x \sqrt{t}\,dt\right]$

The Second Fundamantal Theorem of Calculus

If f is continuous on an open interval containing a constant a, then for every x in the interval,

$$F'(x) = \frac{d}{dx}\left[\int_1^{g(x)} f(t)\,dt\right]$$

$$= f[g(x)] \cdot g'(x)$$

Example: Use the 2nd Fundamental Theorem of Calculus to determine the following:

$$\frac{d}{dx}\left[\int_4^x \cos t^2\,dt\right]$$

$$\frac{d}{dx}\left[\int_x^0 \sqrt[3]{t^2 - t}\,dt\right]$$

$$\frac{d}{dx}\left[\int_{-x}^x t^2\,dt\right]$$

$$\frac{d}{dx}\left[\int_0^{x^2} \sin t\,dt\right]$$

$$\frac{d}{dx}\left[\int_{3x-2}^5 \sqrt[5]{t+4}\,dt\right]$$